IDENTIFICATION OF A SOURCE FUNCTION BY B-SPLINE REGULARIZATION

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We are given a linear second-order differential equation, whose operator is denoted by L. Then

$$Lu(x) = p(x)\frac{d^2u(x)}{dx^2} + q(x)\frac{du(x)}{dx} + r(x)u(x) = f(x) \quad (1)$$

with the boundary conditions

$$u(0) = u_0,\ \frac{du(0)}{dx} = u_{x0};\ u(1) = u_1,\ \frac{du(1)}{dx} = u_{x1}, \quad (2)$$

where $x \in \Omega = [0, 1]$, p(x) is a function that can be continuously differentiated twice, while $p(x) \neq 0$ for all $x \in \Omega$; q(x) is a continuously differentiable function and r(x) is a continuous function. We assume that the function u(x) is measured at certain points $x_i \in \Omega$ $(i = \overline{1, I})$ in the presence of additive stationary noise $\xi(x_i)$ with zero mean and finite variance σ^2. Then

$$u(x_i) = u^*(x_i) + \xi(x_i), \quad (3)$$

where u*(x) is the exact volume of u(x). For simplicity, we assume that the measurements are made on a uniform net with step $h = x_i - x_{i-1}$ $(i = \overline{2, I})$, and the boundary conditions are known exactly.

The problem is to determine the function f(x) from the measured values $u_i = u(x_i)$; this is an irregular inverse problem and involves solving a Fredholm operator equation of the first kind [1]:

$$Af = \int_0^1 G(x, \xi) f(\xi)\, d\xi = u(x), \quad (4)$$

where $G(x, \xi)$ is the Green's function of (1) with the boundary conditions $G(0, \xi) = G(1, \xi) = 0$, $\xi \in [0, 1]$. Problems of this type belong to the class of inverse problems in thermal conduction, where it is necessary to determine the steady-state source or sink distribution by reference to measured temperature profiles.

To determine f(x) we find the quantities $f^0(x)$, $f^1(x)$ and $f^2(x)$, which are determined from the measured values and are estimators for u(x), du/dx and d^2u/dx^2 correspondingly.

We define an extended uniform net with step H, Δ_H^2: $x_{-2} < x_{-1} < x_0 = 0 < \ldots < x_k < \ldots x_K < x_{K+1} < x_{K+2}$. We delete from Δ_H^2 the nodes x_{-2} and x_{K+2}, which gives us the net Δ_H^1, while from Δ_H^1 we selete the nodes x_{-1} and x_{K+1}, which gives us the net Δ_H^0. On Δ_H^2 we define a system of normalized basic splines (B splines) B_k^2 $(k = \overline{-1, K})$, and on Δ_H^1 we define B_k^1 $(k = 0, K)$ and on Δ_H^0 we define B_k^0 $(k = \overline{1, K})$ [2]. The functions $f^2(x)$, $f^1(x)$, $f^0(x)$ are defined in the spaces of B splines of the second, first, and zeroth orders:

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$$f^n(x) = \sum_{k=1-n}^{K} f_k^n B_k^n(x) \approx \frac{d^n u}{dx^n}, \quad n = 0,\ 1,\ 2. \tag{5}$$

We solve these auxiliary problems by using Ritz or Bubnov—Galerkin variational methods [3]. For this purpose, the right and left sides of the expansion are multiplied scalarly by a system of functions $B_k^n(x)$ (n = 0; 1; 2), while the right sides are integrated n times by parts (integration by parts is not performed for n = 0). We get three systems of linear equations:

$$\Gamma_n f^n = u^n,\ n = 0,\ 1,\ 2, \tag{6}$$

where

$$\Gamma_n = \{\gamma_{kl}\} = \left\{ \int_0^1 B_k^n(x)\, B_l^n(x)\, dx \right\},\ f^n = \{f_k^n\},$$

$$u^0 = \{u_k^0\} = \left\{ \int_0^1 \hat{u}(x)\, B_k^0(x)\, dx \right\},$$

$$u^1 = \{u_k^1\} = \left\{ \int_0^1 \hat{u}(x) \frac{dB_k^1}{dx}\, dx + u_1 B_k^1(1) - u_0 B_k^1(0) \right\},$$

$$u^2 = \{u_k^2\} = \left\{ \int_0^1 \hat{u}(x) \frac{d^2 B_k^2(x)}{dx}\, dx + u_{x1} B_k^2(1) - u_{x0} B_k^2(0) - u_1 \frac{dB_k^2(1)}{dx} + u_0 \frac{dB_k^2(0)}{dx} \right\},$$

and $\hat{u}(x)$ in the interpolant of u_i, $i = \overline{1, I}$, k, $l = \overline{1, K}$.

The errors between $f^n(x)$ and $f^{n*}(x)$ are estimated from the continuous norm

$$\varepsilon^n = \|f^n(x) - f^{n*}(x)\| = \max_{x \in \Omega} |f^n(x) - f^{n*}(x)|,\quad n = \overline{0, 2}. \tag{7}$$

The triangle theorem indicates that the error of (7) consists of the sum of the error in solving (6) ε_s^n and the interpolation error (expansion of (5)) ε_I^n (n = 0, 1, 2); without proof we give the bounds for these errors:

$$\varepsilon^0 \leqslant H\omega(f^{0*}) + 2\sigma\sqrt{h/H},$$

$$\varepsilon^1 \leqslant H\omega(f^{1*}) + 2\sigma\|\Gamma_1^{-1}\|\sqrt{h/H^3}\left[\sum_{s=0}^{n}\sum_{r=0}^{n}(-1)^{s+r} C_n^s C_n^r\right]^{1/2}, \tag{8}$$

$$\varepsilon^2 \leqslant 3H\omega(f^{2*}) + 2\sigma\|\Gamma_2^{-1}\|\sqrt{h/H^5}\left[\sum_{s=0}^{n}\sum_{r=0}^{n}(-1)^{s+r} C_n^s C_n^r\right]^{1/2}.$$

The norms of the symmetrical matrices $\|\Gamma_1^{-1}\|$, $\|\Gamma_2^{-1}\|$ conform with the norm of (7) and can be estimated while solving (6). The quantity $\omega(f^n)$ is the continuity modulus for the function $f^n(x)$ over an interval of length Ω [2]. We use (8) and (1) to write

$$\varepsilon = \|f(x) - f^*(x)\| \leqslant \|p(x)\|\,\varepsilon^2 + \|q(x)\|\,\varepsilon^1 + \|r(x)\|\,\varepsilon^0. \tag{9}$$

With given h, σ, and $\omega(f^{n*})$ (n = 0, 1, 2) the step H may be found by minimizing the estimator $\omega(H)$ with respect to the desired quantity. Therefore, the step H conforming to the errors of the initial data can be determined from

$$\partial\varepsilon(H)/\partial H = 0, \tag{10}$$

which is a necessary and sufficient condition for a minimum in the upper value of the bound $\varepsilon(H)$ in view of the strict convexity with respect to H.

The regularization parameter in this algorithm is the step H for a given h. This defines the dimensions of the interpolation subspaces for the interpolation of $f^n(x)$ ($n = 0, 1, 2$); the regularization itself is performed by choosing suitable interpolation subspaces with bases $B_k^0(x)$, $B_k^1(x)$, $B_k^2(x)$ and the dimensions of these. Therefore, the method is best called B-spline regularization.

The solution in that case is obtained as a piecewise-smooth one with discontinuities of the first kind, which define a piecewise-constant function $f^0(x)$ that approximates to $\hat{u}(x)$. To improve the smoothness of the solution, perhaps while even gaining accuracy, we can use a smoothing polynomial or spline of higher order as $f^0(x)$ without particular attention to the degree of accuracy of the smoothing.

The method has been tested for the case where $p(x) = 1$, $q(x) = 0$, $r(x) = 0$ in (1), i.e., to define the second derivative of the measured function. It was assumed that $\sigma = 0.1$, i.e., the initial data were corrupted by noise to 10% of the mean exact value. To determine H, (10) was solved by Newton's iteration method. The continuity moduli $\omega(f^{0*})$, $\omega(f^{1*})$, $\omega(f^{2*})$ were specified. With a given h, the initial approximation H^0 was chosen from the condition $H/h \geqslant 5$. The step H was determined by solving the equation

$$5\sigma\,\|\Gamma_2^{-1}\|\left[\sum_{s=0}^{n}\sum_{r=0}^{n}(-1)^{s+r}C_n^s C_n^r\right]^{1/2}\sqrt{h/H^7} = 3\omega(f^{2*}). \tag{11}$$

The numerical simulation was performed for a monotonically increasing polynomial of third degree and a sinusoidal function. The length of the interval Ω was equal to the period of the sine wave. The variation in the polynomial within the interval Ω was large, ~10,000, while that of the sinusoidal function was of the order of one. Practical calculations showed that it is laborious to solve this problem with smoothing cubic splines for functions with large variations. It is also difficult to select weighting factors to construct the smoothing curve that will provide a good smoothing over the entire range. Usually, these are selected by means of a numerical experiment, i.e., one selects heuristically a curve providing good smoothing. Then the resulting spline is differentiated twice. The method described here is free from this deficiency.

The errors in the second derivatives indicated by the continuous norm did not exceed 18% with 10% errors in the initial data, 5% errors in the boundary conditions, and numbers of points from 20 to 30.

NOTATION

L, differential operator; $u(x)$, $f(x)$, measured and unknown functions; $p(x)$, $q(x)$, $r(x)$, coefficients in the differential equation; Ω, region of function determination; $B_k^n(x)$, k-th basic spline of order n; ε_s, error in solution of the linear equation system; ε_I, interpolation error; $\omega_H(f)$, modulus of the discontinuity of $f(x)$ in the interval of length H; h, measuring interval step; σ, standard deviation of measurement error.

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